Non-Linear Constitutive Equations for Sand under Cyclic Loading and Application to Geotechnical Earthquake Engineering

S. A. Savidis, D. Aubram, F. Rackwitz, W. Schepers

Soil Mechanics and Foundation Engineering Division
Technical University of Berlin, Germany

Abstract

Two advanced constitutive relations are introduced to model the strong non-linear material behavior of the sandy soil continuum. The first sand model, the CSSA-Model, is based on the critical state concept as well as on the bounding surface plasticity framework and the concept of generalized plasticity. It uses a state dependent dilatancy approach as a special feature. The second constitutive model is assigned to the hypoplastic framework. The material models were implemented into the implicit FE-code ANSYS which has been used for the analyses of element tests. Both, the CSSA-Model and the hypoplastic model, are capable to describe the particular sand behavior over a wide range of density and stress states with only one set of material parameters for each model. Especially the critical state sand model predicts the material response under cyclic loading conditions very satisfactorily. Hence, the CSSA-Model is particularly suitable for the application to the one-dimensional shear wave propagation.

INTRODUCTION

The realistic simulation of shear wave propagation in sands requires an appropriate constitutive equation for the granular material. Because of the fact that the mechanical behavior of sand depends on the density state, the stress state and the stress history as well as on the drainage situation, the continuum mechanical approach is rather complicated. Due to shear loading under the same stress state, the volume of a dense sand sample dilates, whereas a loose sand contracts. Irrespective of the current void ratio, sand exhibits an isochoric response if shearing continues, which is well-known as critical state. Additionally, sand is exposed to liquefaction under cyclic loading with undrained conditions.

Classical soil models treat the diverse appearances of sand behavior as different materials, although it is still the same sand. In the following two different models are presented. Both simulate the mechanical behavior of sand with a single set of parameters quite well, by taking the density as a state variable into account.

THE CSSA-MODEL

Model Framework

To account for the above mentioned facts of sand behavior, the critical state sand model (CSSA-Model) developed by Li (2000) incorporates a state dependent dilatancy. There are three concepts in the history of material modelling which affected the development of the elastic-plastic CSSA-Model. The well known Critical State framework (Roscoe et al. 1958) is effectively the basic idea. To model the response of sand under cyclic loading, the Bounding Surface Plasticity concept developed by Dafalias (1986) has been implemented. The Generalized Plasticity framework extended by Pastor et al. (1990) circumvents the difficulty to formulate the hardening and softening functions precisely by describing only the rate of these functions, also referred to as the plastic modulus.

Consider three specimen of the same sand with the same loading history and starting from the same stress state, but with different densities. If a shear loading increment from the same stress ratio \( \eta = q / p' \) is applied, the loose
sand contracts and the dense sand dilates, like show in the triaxial \( q - p' \) plane in Fig 1, where \( q = \sigma'_1 - \sigma'_3 \), \( p' = (\sigma'_1 + 2\sigma'_3)/3 \) and \( \sigma'_3 \) are the axial and the radial effective stress, respectively. In addition, medium-dense and dense sand passes a so-called phase transformation state (Ishihara et al. 1975, Li and Dafalias 2000). At this point the dilatancy \( D \) is equal to zero, although the sand is not at a critical state.

Li (1998) has shown that there exists a linear representation of the critical state line (CSL) in the \( e - \left(\frac{p'}{p_a}\right)^{\xi} \) plane, where \( e \) is the void ratio, \( p_a \) is the atmospheric pressure and \( \xi \) is a material parameter (Fig 2). The internal state parameter \( \psi \) can be defined from Fig 2 as

\[
\psi = e - e_c = e - \left[ \epsilon_1 - \lambda_c \left( \frac{p'}{p_a} \right)^{\xi} \right],
\]

where \( \epsilon_1 \) and \( \lambda_c \) are material parameters.

One of the main features of the CSSA-Model is the state dependent dilatancy, due to the fact that the dilatancy for sand depends on the stress ratio \( \eta \) as well as on the void ratio \( e \).

The dilatancy function \( D_1 \) for the first bounding surface (Fig 3) has the form

\[
D_1 = \frac{d_1}{M_c g(\theta)} \left[ M_c g(\theta) e^{\psi_0} \sqrt{\rho_1 - R} \right],
\]

where \( d_1, M_c, m \) are material parameters. The CSSA-Model includes 19 parameters - 15 material parameters determined from drained and undrained triaxial tests plus 4 default model constants. \( \rho_1, \rho_1 \) are the mapping distances referred to the Bounding Surface Plasticity framework. The function \( g(\theta) \), where \( \theta \) is the Lode angle, establishes the shape of the first bounding surface and \( R \) is an invariant of the stress ratio tensor \( r_{ij} = s_{ij} / p' \), where \( s_{ij} \) is the deviatoric stress tensor.
Numerical Implementation

Because most of the geotechnical problems can only be solved numerically, a numerical representation of the soil model is needed. In a finite element code, the iteration of the global equilibrium is often done by a Newton-Raphson scheme (Fig 4). The local stress update, also referred to as the stress point algorithm, can be derived by an Euler method.

Fig 4: Newton-Raphson iteration of the global equilibrium

There are two groups of stress point algorithms. The explicit stress point algorithms are formulations using well known terms at the beginning of an increment, like the forward Euler scheme. Implicit stress point algorithms are formulations using terms at the end of an increment, like the backward Euler scheme. As a matter of fact, implicit algorithms usually require a specific formulation of the yield surface. Therefore, the application to generalized plasticity models, especially to the CSSA-Model, needs further investigations. The CSSA-Model has been implemented into the general purpose FE-code ANSYS using the material models interface usermat (Rackwitz (2003), ANSYS Inc. (2000)).

Numerical Simulation of Element Tests

The implementation of the CSSA-Model in ANSYS has been tested by simulation of element tests (Rackwitz (2003), Aubram (2004)). For the following numerical simulation of element tests the material parameters for Toyoura sand according to Li (2002) has been used. Fig 5 shows the simulation of an undrained triaxial compression test with Toyoura sand and four different initial confining pressures in the $q - p'$ stress plane (Fig 5 top) and stress – axial strain plane (Fig 5 bottom).

Fig 5: Simulation of undrained triaxial compression tests, Toyoura sand, $D_r = 65\%$

Simulations of the experimental results of monotonic triaxial tests using the CSSA-Model are quite well, which has been presented by Li and Dafalias (2000). In contrast to that, the results of the simulation of a cyclic undrained triaxial test differ slightly, dependent on the size of the integration increments, as shown in the
Since the explicit substepping algorithm is only exact if the increments are infinite small, the calculation error accumulates along with the number of load steps. Notwithstanding this fact, the CSSA-Model again provides good results. If the number of loading cycles increases, starting from isotropic consolidation, the effective mean normal stress decreases at constant density as a result of a rearrangement of the particles. In the region where \( p' \) vanishes, the sand begins to liquefy, staying within the bounding surfaces.

The results show that the CSSA-Model simulates the behavior of sand under monotonic and cyclic loading over a wide range of densities and stress states using a single set of parameters. Nevertheless, the results depend on the implemented stress point algorithm and react sensitively to the size of the calculation increments. It seems reasonable to implement other stress point algorithms, especially implicit algorithms in further investigations and to check and modify the formulation of the plastic moduli of the two bounding surfaces to enhance the model response to boundary value problems.

**THE HYPOPLASTIC MODEL**

**Model Framework**

The framework of hypoplasticity according to Kolymbas (2000), and Kolymbas et al. (1995, 1993) is an alternative and completely different approach compared to “classical” elastoplasticity. Hypoplasticity does not distinguish between elastic and plastic strains and it neither uses any yield or plastic potential surface nor normality and consistency rules. The name “hypoplastic” was taken from the relation between hypoplasticity and classical (elasto)plasticity on one side and hypoelasticity and elasticity on the other side. Both theories
with the prefix “hypo” do not require the use of a potential (Kolymbas et al. (1995, 1993)). The general constitutive equation for that class of hypoplastic models is of the form

\[
\dot{T} = h(T, D, e),
\]

where \( \dot{T} \) is the objective stress rate tensor as a functional \( h \) of the Cauchy stress tensor \( T \), the deformation rate tensor \( D \) and the void ratio \( e \).

The particular hypoplastic model according to von Wolffersdorff (1996), which has been used here, has the form

\[
\dot{T} = f_b f_e \frac{1}{tr(\dot{T})} \left[ F^2 D + a^2 \dot{T} tr(\dot{T} D) \right] + f_d a F \left( \dot{T} + \dot{T}^* \right) \|D\|,
\]

with \( \dot{T} = T/\text{tr}T \) and \( \dot{T}^* = \dot{T} - \frac{1}{3} I \), where \( I \) is the 2nd order unit tensor and \( \text{tr} \) is the trace of a 2nd order tensor.

\[
a = \sqrt{\frac{3 \left(3 - \sin \phi_c\right)}{8 \sin \phi_c}}
\]

\[
F = \sqrt{\frac{1}{8}} \tan^2 \psi + \frac{2 - \tan^2 \psi}{2 + \sqrt{2} \tan \psi \cos 3\Theta} - \frac{1}{2\sqrt{2}} \tan \psi
\]

\[
\cos 3\Theta = -\sqrt{6}\left[ \frac{\text{tr}(\dot{T}^*)}{\text{tr}(\dot{T}^*)} \right]^{\frac{3}{2}} \tan \psi = \sqrt{3} \|\dot{T}^*\|
\]

The functions \( f_e \), \( f_d \) and \( f_b \) take into account the influence of density and stress state on the material behavior.

\[
f_e = \left( \frac{e_c}{e} \right)^\beta, \quad f_d = \left( \frac{e - e_d}{e_c - e_d} \right)^\alpha
\]

\[
f_b = \frac{h_n \left( 1 + e_i \right)}{n \left( e_i \right)^\beta \left( e_c/e_0 \right)^{\alpha \left( trT \right)^{1-n}}} - \frac{trT}{h_n}
\]

That hypoplastic model uses 8 material parameters - the critical friction angle \( \phi_c \), the granular hardness \( h_i \), the limiting void ratios \( e_{c0}, e_{d0}, e_0 \) and the exponents \( n, \alpha, \beta \) - all together to be determined mainly from standard soil mechanical laboratory tests (Herle and Gudehus (1999)).

The hypoplastic model for sand (von Wolffersdorff (1996)) was extended by Niemunis and Herle (1997) in order to simulate the mechanical behaviour under cyclic loading conditions. By including a so-called “intergranular strain”, which serves as an additional state variable, the stiffness is modified dependant on the direction of the previous strain increment.

Numerical Implementation

In general the numerical implementation of hypoplastic constitutive equations follows the same procedures as for classical elastoplastic models. But because of the fact that there are no surfaces explicitly defined, there is no need to develop the numerical solution in respect to any such yield or potential surface. A standard numerical integration scheme, as described above, can be applied instead, to integrate the constitutive equations for each time step during the numerical solution process. That makes the numerical implementation of hypoplastic models much easier and leads to quite robust numerical behavior (Rackwitz (2003), Kolymbas (2000)).

Numerical Simulation of Element Tests

The above described hypoplastic soil model by von Wolffersdorff (1996) was also implemented into the general purpose FE-Program ANSYS and the parameters for that constitutive model were evaluated for Berlin sand (Rackwitz (2003)). The data of triaxial compression tests with Berlin sand can now be used to check the quality of the used hypoplastic constitutive equation.

The comparison of laboratory and numerical results from drained triaxial compression tests at a confining cell pressure \( p_0 = 500 \text{ kPa} \) is shown in Fig 8. Starting from three different initial relative densities \( D_n \), ranging from loose to dense, the stress-strain paths of all tests reach a joint critical state after monotonic shearing up to about 21% axial strain (Fig 8 top). The
The hypoplastic model also represents typical volumetric behavior of sand (Fig 8 bottom). Dilatancy of initially medium dense to dense sand is well predicted. The results for initially loose samples ($D_r = 13\%$) are less accurate.

OUTLOOK

The lecture continues with the application of different approaches to the one-dimensional shear wave propagation. The computer program SHAKE91 will be used for the calculations including an equivalent linear viscoelastic model and the FE-code ANSYS for the truly non-linear CSSA-Model. Since pure (bending-free) shearing can not be handled by the majority of one-dimensional finite elements, plane distorted quadrilateral elements in combination with special boundary conditions will be used.

The response of both approaches undergoing base excitation will be compared. In contrast to the equivalent linear model, the CSSA-Model, due to its concept of state dependent dilatancy, will show some density development and vertical displacements of the ground surface under drained conditions. Under undrained conditions, a pore pressure build up in the soil profile may occur, followed by a decrease of the effective stresses. Thus, conclusions can be drawn about the cyclic mobility of the sand deposit or the exposure to liquefaction. Finally, these results will point out the capabilities of advanced constitutive equations for soils in earthquake engineering.

REFERENCES

ANSYS Inc. (2000) Guide to ANSYS 5.6 User Programmable Features


